Int. J. Heat Mass Transfer. Vol. 9, pp. 1455–1461. Pergamon Press 1966. Printed in Great Britain

DATA REDUCTION USING THE LEAST SQUARE METHOD

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(*Received* 10 *March* 1966 *and in a revised form* 1 *April* 1966)

Abstract—In many cases an unknown function $T(t)$ is approximated by a number of observations $T_i(t_i)$ and the slope $\partial T/\partial t$ is needed. The slope is often calculated by fitting a polynomial through a group of the $T_i(t_i)$ points using the least square method. In the present paper the error introduced into the slope is determined for some cases which are representative for experiments using the transient heat-transfer technique. The results can either be used when new experiments are planned or when the best combination of parameters (number of points approximated by the polynomial, degree of the polynomial) has to be determined for the reduction of existing data.

It was also found that, within the framework of assumptions made, an even degree polynomial gives exactly the same slope as the odd one with the next lower degree. Therefore, only odd degree polynomials should be used.

NOMENCLATURE

a,	coefficients in the polynomial;
c,	specific heat of wall material;
d,	wall thickness;
n,	degree of polynomial;
q,	heat-transfer rate;
T,	true wall temperature;
T_i,	measured wall temperature;
T_r,	recovery temperature;
t,	true time;
t_i,	measured time;
ΔT_{rms},	root-mean-square error of $T - T_i$;
ΔT,	increase of the temperature of a thermocouple between two readings;
Δt,	time between two readings;
α,	heat-transfer coefficient;
δ,	non-dimensional error in $\partial T/\partial t$;
ϵ,	non-dimensional r.m.s. error of the temperature;
θ,	polynomial approximating $T_i(t_i)$;
$2\nu + 1$,	number of points $T_i(t_i)$ approximated by polynomial;
ρ,	density of wall material;
τ,	time constant.

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INTRODUCTION

THE TRANSIENT technique is often used to determine the heat transferred from a flowing medium to a body. Usually a thin-walled model is built and the wall is instrumented with thermocouples. If heat is transferred to the wall, the wall temperature T increases. A simple energy balance for a wall element connects the heat flow rate q with the slope of the temperature–time curve. The relation

$$q = \alpha(T_r - T) = \rho c d \frac{\partial T}{\partial t} + \text{(heat loss by conduction)} + \text{(heat loss by radiation)}. \quad (1)$$

Here, α is the heat-transfer coefficient, T_r the recovery temperature, ρ and c are the density and the specific heat of the wall material respectively and d is the thickness of the wall.

If a digital recorder is used it is usual to record the temperatures $T_i(t_i)$ with constant time intervals. Using the least square method a polynomial of the form $\theta = a_0 + a_1 t + \dots a_n t^n$ is then fitted through a number of these $[T_i(t_i)]$ points and the slope, $\partial\theta/\partial t$, is determined near the middle of the interval.

In many cases, the heat losses by conduction and radiation [in equation (1)] can be neglected and the change in wall temperature during a run is so small that α, T_r, ρ and c can be assumed to be constant. In this case equation (1) can be integrated. The result is

$$\frac{T - T_r}{(T - T_r)_{t=0}} = \exp\left[-(\alpha t/\rho c d)\right]. \qquad (2)$$

Often, equation (2) is a good approximation for that part of the $T(t)$ relation that is approximated by the polynomial. If not, the corrections for conduction and radiation are large and dominate the accuracy any way.

The number of points and the degree of the polynomial used are usually determined "by eye". The present note gives a possibility to determine the best combination for a given experimental arrangement (see postscript).

RESULTS

The theory is based on the following assumptions which are not too restricting for typical heat-transfer experiments.

(1) Only one variable (say T) is subject to an error.
(2) This error $(T_i - T)$ is random and has either a normal distribution or the same probability that it is $+(\sqrt{\tfrac{3}{2}})\,\Delta T_{\text{rms}}$, 0, or $-(\sqrt{\tfrac{3}{2}})\,\Delta T_{\text{rms}}$.
(3) The interval Δt between the readings of the other variable $(t_{i+1} - t_i)$ is constant.
(4a) The relation $T(t)$ is of the form $T \sim e^{-t}$ [equation (2)], or
(4b) The effect of the curvature of the $T(t)$ relation can be neglected for the t-interval approximated by the polynomial.
(5) The slopes $\partial\theta/\partial t$ and $\partial T/\partial t$ are determined in the middle of the t-interval.

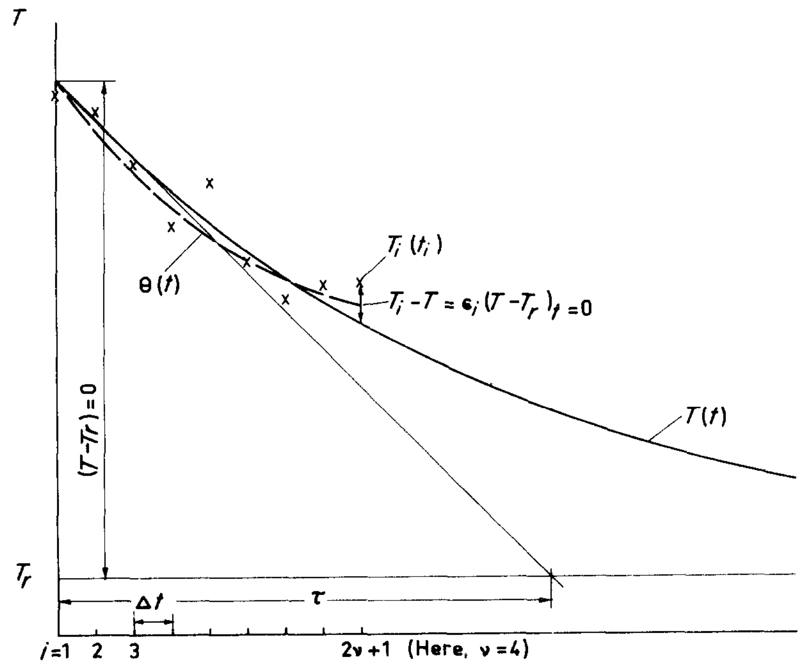

FIG. 1. Sketch illustrating the method and the nomenclature.
$T(t)$ = true temperature–time relation;
$T_i(t_i)$ = $2\nu + 1$ measured data points;
$\theta(t)$ = Polynomial fitted through the $2\nu + 1$ points.

If assumptions 1, 2, 3, 4a and 5 are satisfied the root-mean-square (r.m.s.) error δ of the slope can be given in the form

$$\delta = fn(\Delta t/\tau, \epsilon, 2\nu + 1, n). \tag{3}$$

Here the following definitions (see Fig. 1) are used

$$\delta = \left(\frac{\partial T/\partial t - \partial\theta/\partial t}{\partial T/\partial t}\right)_{\text{rms-value for many repeated runs}} \tag{4}$$

Δt = time interval between two subsequent readings of a thermocouple

$$\tau = \frac{\rho c d}{\alpha} = -\left(\frac{\dot{T} - T_r}{\partial T/\partial t}\right)_{t=0} \tag{5}$$

$$\epsilon = \left|\frac{\Delta T_{\text{rms}}}{(T - T_r)_{t=0}}\right| \tag{6}$$

ΔT_{rms} = r.m.s.-value of many $T_i - T$ values
$2\nu + 1$ = number of points $T_i(t_i)$ approximated with the polynomial
n = degree of polynomial.

The relation given by equation (3) was determined in two different ways. In a simple-minded approach, an IBM 7090 computer was programmed to superimpose random errors (distributed according to the two probability distributions described above) upon the exact $T(t)$ relation given by equation (2), resulting in groups of $2\nu + 1$ points $T_i(t_i)$. For each of these groups a figure for δ and ϵ was calculated. This was repeated with 270 groups and approximate figures for δ and ϵ were calculated with equations (4) and (6). As this method required excessive computer time, it was only used to check the approach described below and to obtain information about the effect of the probability distribution of the error $T_i - T$. Within the accuracy of the results, no systematic difference could be detected for the two probability distributions used.

In the second approach the error δ was determined directly with a statistical method described in reference [1]. The result is

$$\delta = \sqrt{\left\{\delta_0^2 + \left(\epsilon\frac{\tau}{\Delta t}\exp\left[\nu\,\Delta t/\tau\right]\right)^2 \sum_{i=1}^{2\nu+1}\left[\sum_{l=0}^{n} c_l(i-1)^l\right]^2\right\}}. \tag{7}$$

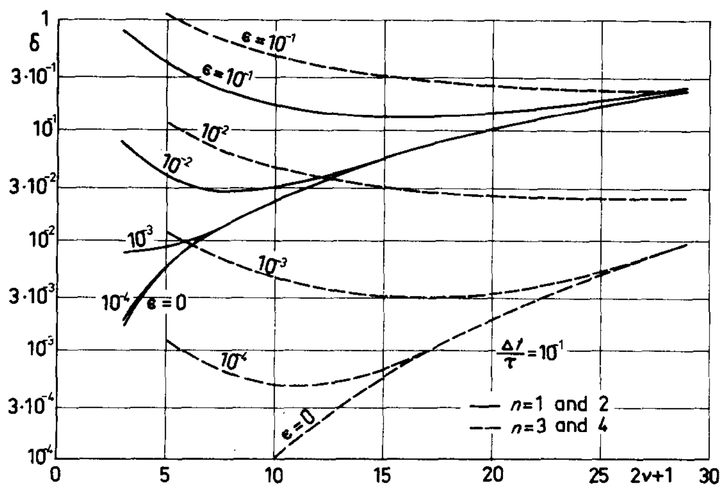

FIG. 2. Relative error δ in the slope of the $T(t)$-curve as a function of the degree n of the polynomial used, of the error ϵ in the temperature reading and of the number $2\nu + 1$ points used.

Apparently δ_0 is the error for vanishing ϵ, that is the error introduced by approximating the exponential in equation (2) with a polynomial. The c_i are complicated functions which contain integers and the constant Δt. As they are independent of the T_i they can be tabulated once and for all. This was done on an IBM 7090 computer. The results are shown in Figs. 2–4. They can be used for both distribution functions specified in assumption (2). The most important result is that δ is exactly the same for $n = 1$ and 2 if assumptions 1, 3 and 5 are satisfied. The same is true for $n = 3$ and 4, 5 and 6, etc. Only *odd* polynomials should be used therefore, as

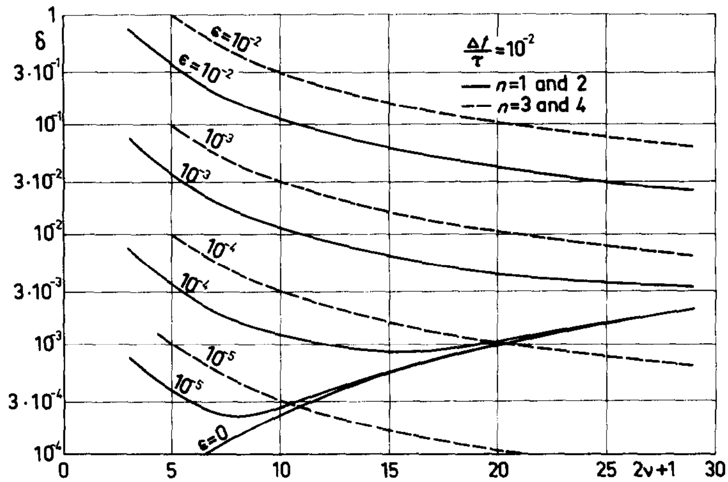

FIG. 3. Relative error δ in the slope of the $T(t)$-curve as a function of the degree n of the polynomial used, of the error ϵ in the temperature reading and of the number $2\nu + 1$ points used.

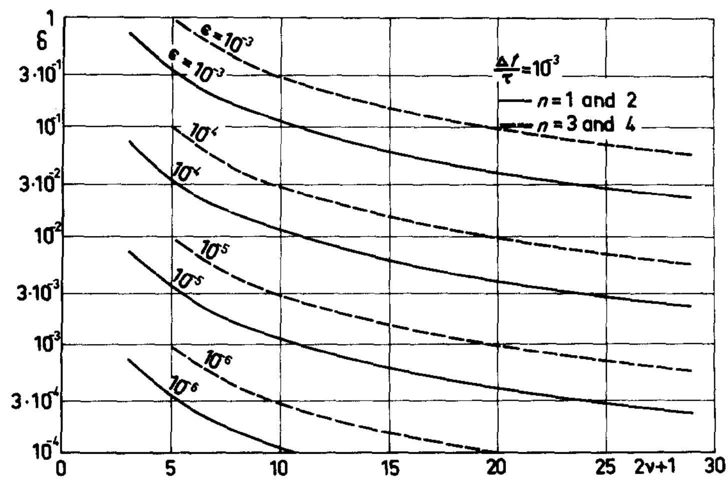

FIG. 4. Relative error δ in the slope of the $T(t)$-curve as a function of the degree n of the polynomial used, of the error ϵ in the temperature reading and of the number $2\nu + 1$ points used.

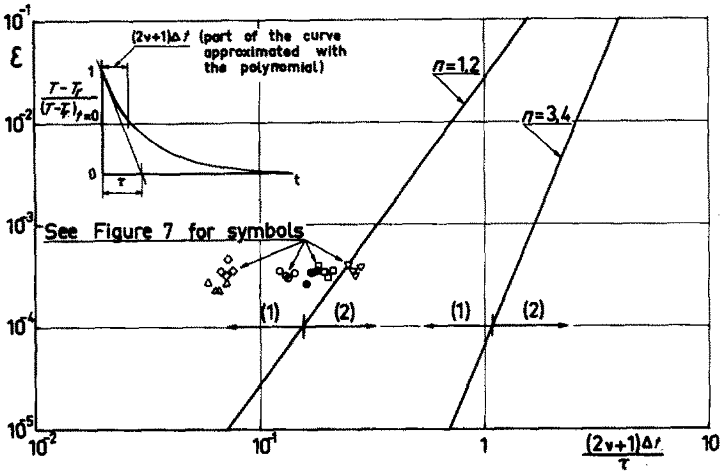

FIG. 5. Relation between the error ϵ in the temperature reading and the relative length of the $T(t)$-curve for minimum error δ of the slope. In range 1 the error is mainly due to scatter of the temperature readings, in range 2 the error is mainly due to the curvature of the $T(t)$ relation. ↔Along this line, δ has its minimum.

the even ones with the next higher degree do not increase the accuracy. They only lead to increased complexity of the data reduction. If a small part of the $T(t)$—relation is approximated with the polynomial, the effect of the temperature scatter dominates, if a large part is used, the effect of the curvature dominates. Between these two ranges, δ has its minimum. The relation between the scatter ϵ and the part of the curve used $(2\nu + 1)\ \Delta t/\tau$ for minimum δ is given in Fig. 5.

If assumption (4a) can be replaced by (4b), equation (3) can be written as

$$\delta = \frac{\Delta T_{\text{rms}}}{\Delta T} fn(2\nu + 1, n). \tag{8}$$

Here, $\Delta T = |\partial T/\partial t|\ \Delta t$ is the increase of the temperature of a thermocouple between two subsequent readings. Results for $n = 1, 2$ and $3, 4$ are given in Fig. 6.

In order to determine whether the curvature can be neglected in an actual case, figures for ϵ, ν, and $\Delta t/\tau$ should be inserted into Fig. 5. If they determine a point well in range 1 of this

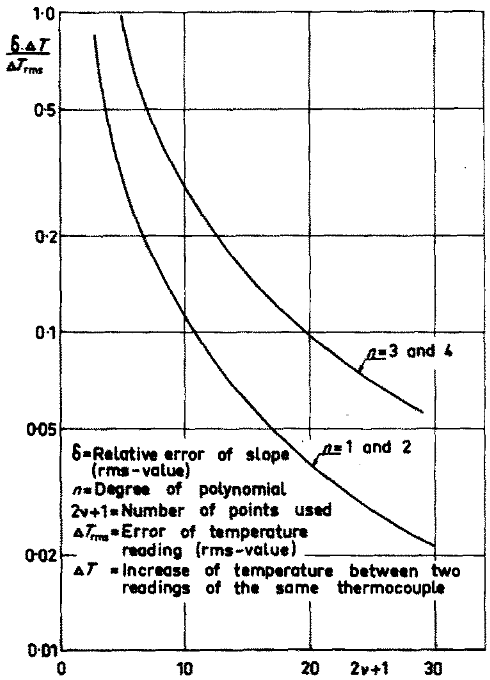

FIG. 6. Relation between the relative error δ of the $T(t)$ slope, the degree of the polynomial n and the number $2\nu + 1$ of points, if the curvature of the $T(t)$ relation can be neglected.

figure and if the curvature of the experimental curve is equal to or less than the curvature of a comparable exponential, equation (8) may be used.

COMPARISON WITH AN EXPERIMENT

The results of the present investigation were applied to the reduction of the experimental data described in reference [2]. In this experiment the heat transfer to a thin-walled cone was measured by determining the slope of the $\theta(t)$ relation where θ is the polynomial approximating the $T_i(t_i)$. As $T(t)$ is unknown ΔT_{rms} was approximated by $(T_i - \theta)_{rms}$. The value of δ was determined by calculating the scatter of the Stanton numbers reduced at different times

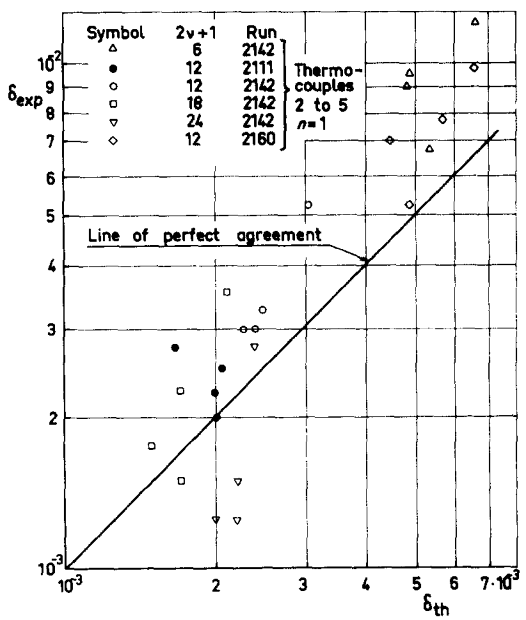

FIG. 7. Comparison of the calculated error δ_{th} with the error δ_{exp} determined from an experiment.

by comparing them with a second degree polynomial. The results are shown in Fig. 7 and the agreement is seen to be satisfactory.

CONCLUSIONS

If transient heat-transfer experiments are reduced, the present results can be used when planning a new experiment and when reducing the data.

If a new experiment is planned, the wind tunnel and the recording system are usually given and ΔT_{rms} and $(T - T_r)_{t=0}$ are known or can be

the wall thickness of the model (changing τ) and the number of thermocouples (changing Δt). With the aid of Figs. 2–6 the most satisfactory combination between these parameters and the error δ can be found.

If existing data are to be reduced, $\Delta t/\tau$ and ϵ can be determined and Figs. 2–6 can be used to get a compromise between accuracy and required computer time.

Finally it is pointed out that in this case even degree polynomials give exactly the same result for the slope as odd ones with the next lower degree. Reducing the slopes to Stanton numbers, ρ and c are needed. As they are weak functions of T and as $\theta(\nu\ \Delta t)$ is different for $n = 1$ and 2, the Stanton numbers will be slightly changed if $n = 1$ is replaced by $n = 2$. However, the effect is usually hardly observable.

REFERENCES

1. H. THOMANN and G. LINDSJÖ, Optimisation of the reduction of transient heat transfer data. The Aeronautical Research Institute of Sweden (FFA), Report 106 (1966).
2. H. THOMANN, Model injection device for a hypersonic wind tunnel. The Aeronautical Research Institute of Sweden (FFA), Memo 39 (1965).

POSTSCRIPT

During the publication of the present investigation a related one with the title "Random errors of derivatives obtained from least square approximation to empirical functions" was published by Hans C. Joksch in *SIAM Rev.* **1** (1), (1966). This latter work gives the error of the first and second derivative in the whole interval approximated by the polynomial and not only in the middle as the present paper does. The present paper, on the other hand, takes into account the effect of the curvature of a $T(t)$ relation typical for transient heat transfer measurements. The expression $\delta\Delta T/\Delta T_{rms}$ in Fig. 6 is identical with Joksch's $\sqrt{(R_1^1)}$ and $\sqrt{(R_3^1)}$ in Tables 2 and 3.

Résumé—Dans de nombreux cas, on approche une fonction inconnue $T(t)$ par un certain nombre d'observations $T_i(t_i)$ et l'on a besoin de la pente $\partial T/\partial t$. La pente est souvent calculée en ajustant un polynôme à travers un groupe de points $T_i(t_i)$ à l'aide de la méthode des moindres carrés. Dans cet article, l'erreur introduite sur la pente est déterminée pour quelques cas qui représentent des expériences où l'on emploie la technique du transport de chaleur transitoire. Les résultats peuvent être utilisés soit lorsque de nouvelles expériences sont projetées, soit lorsque la meilleure combinaison de paramètres (nombre de points approchés par le polynôme, degré du polynôme) doit être déterminée pour la représentation des données existantes.

On a également trouvé que, dans le cadre des hypothèses faites, un polynôme de degré pair donne exactement la même pente que le polynôme impair de degré immédiatement inférieur. Donc, seuls des polynômes de degré impair devraient être utilisés.

Zusammenfassung—In vielen Fällen wird eine zunächst unbekannte Funktion $T(t)$ durch eine Anzahl von Messwerte $T_i(t_i)$ approximiert und es interessiert die Ableitung $\partial T/\partial t$. Häufig ermittelt man die Ableitung so, dass unter Verwendung der Methode der kleinsten Quadrate ein, den Messwerten $T_i(t_i)$ angepasstes Polynom aufgestellt wird. In der vorliegenden Arbeit wird der Fehler bei der Bestimmung der Ableitung für einige Fälle ermittelt, die repräsentative für Experimente gelten können, in denen von den Methoden der instationären Wärmeübertragung Gebrauch gemacht wird. Die Ergebnisse können einmal bei der Planung neuer Experimente Verwendung finden, zum anderen lässt sich die günstigste Kombination von Parametern (Anzahl der durch das Polynom approximierten Punkte, Grad des Polynoms) finden, wenn eine Reduktion vorliegender Werte angestrebt wird.

Ferner ergab sich, dass im Rahmen der getroffenen Annahmen ein Polynom geraden Grades den gleichen Wert für die Ableitung liefert, wie das Polynom vom nächst niedrigeren, ungeraden, Grade. Es sollten deshalb nur Polynome von ungeradem Grade verwendet werden.

Аннотация—Во многих случаях неизвестная функция $T(t)$ аппроксимируется рядом опытных точек $T_i(t_i)$, что необходимо для получения наклона кривой $\partial T/\partial t$. Зачастую наклон рассчитывается с помощью полиноминалного приближения к точкам $T_i(t_i)$ по методу наименьших квадратов. В данной работе определяется ошибка наклона кривой для некоторых случаев, характерных для нестационарных методов экспериментов. Полученные результаты можно использовать при планировании экспериментов, а также при определении оптимальной комбинации параметров, обобщающих полученные данные (число точек, аппроксимируемых полиномом, степень полинома).

Установлено, также, что при принятых допущениях полиномы четной степени дают те же результаты, что и полиномы предыдущей нечетной степени. Следовательно, целесообразно использование толбко полиномов нечетной степени.